QUIZ 4 - CALCULUS 2 (2020/12/17)

1. True or False Questions. Mark "O" before correct statements and "X" before incorrect statements.

   (a) (1 pt) _X_ The improper integral $\int_{-1}^{1} \frac{1}{x^3}\,dx$ is zero because $\frac{1}{x^3}$ is an odd function.

   (b) (1 pt) _O_ The improper integral $\int_{1}^{\infty} \frac{dx}{x\sqrt{x+1}}$ is convergent.

   (c) (1 pt) _X_ The improper integral $\int_{0}^{1} \frac{dx}{x\sqrt{x+1}}$ is convergent.

2. Compute the following integrals.

   (a) (5 pts) $\int \frac{x-3}{x^3-x^2+x-1}\,dx$

   Solution:
   $\frac{x-3}{x^3-x^2+x-1} = \frac{-1}{x-1} + \frac{x+2}{x^2+1}.$
   (1 pt for the correct form of partial fractions. 1 pt for correct constants).
   Hence $\int \frac{x-3}{x^3-x^2+x-1}\,dx = -\ln|x-1| + \frac{1}{2}\ln(x^2+1) + 2\arctan(x) + C.$(1 pt for each integration.)

   (b) (6 pts) $\int \frac{1}{2\sqrt{x-5}+x}\,dx$ (Hint: Try the substitution $u = \sqrt{x-5}$.)

   Solution:
   $\int \frac{1}{2\sqrt{x-5}+x}\,dx = \int \frac{2u}{u^2+2u+5}\,du$ (2 pts)
   $= \int \frac{2u}{(u+1)^2+4}\,du \underset{y=u+1}{=} \int \frac{2y-2}{y^2+4}\,dy$ (1 pt)
   $= \ln(y^2+4) - \arctan(\frac{y}{2}) + C$ (2 pts)
   $= \ln|x+2\sqrt{x-5}| - \arctan(\frac{\sqrt{x-5}+1}{2}) + C.$ (1 pt)

   (c) (6 pts) $\int_0^{\infty} \frac{1}{(1+x^2)^2}dx$

   Solution:
   $\int \frac{1}{(1+x^2)^2}\,dx \underset{x=\tan\theta, -\frac{\pi}{2}<\theta<\frac{\pi}{2}}{=} \int \frac{\sec^2\theta}{(1+\tan^2\theta)^2}\,d\theta = \int \cos^2\theta\,d\theta$ (1 pt)
   $= \int \frac{1+\cos 2\theta}{2}\,d\theta = \frac{\theta}{2} + \frac{\sin 2\theta}{4} + C$ (2 pt)
   $= \frac{1}{2}(\arctan x + \frac{x}{1+x^2}) + C.$ (1 pt)
   By the definition, $\int_0^{\infty} \frac{1}{(1+x^2)^2}\,dx = \lim_{t\to\infty} \int_0^{t} \frac{1}{(1+x^2)^2}\,dx = \lim_{t\to\infty} \frac{1}{2}(\arctan t + \frac{t}{1+t^2})$ (1 pt)

$= \frac{\pi}{4}$ (1 pt).

Another solution, $\int \frac{1}{(1+x^2)^2}\,dx = \int \frac{1+x^2}{(1+x^2)^2} - \frac{x^2}{(1+x^2)^2}\,dx = \int \frac{1}{1+x^2} - x \cdot \frac{x}{(1+x^2)^2}\,dx$

$= \arctan x - [x \cdot (-\frac{1}{2}\frac{1}{1+x^2}) + \frac{1}{2}\int \frac{1}{1+x^2}\,dx] = \frac{1}{2}(\arctan x + \frac{x}{1+x^2}) + C.$ (4 pts)

By the definition, $\int_0^\infty \frac{1}{(1+x^2)^2}\,dx = \lim_{t\to\infty} \int_0^t \frac{1}{(1+x^2)^2}\,dx = \lim_{t\to\infty} \frac{1}{2}(\arctan t + \frac{t}{1+t^2})$ (1 pt)

$= \frac{\pi}{4}$ (1 pt).